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ex) Reparametrize  $\vec{r}(t) = \langle 3\sin(t), 2t, 3\cos(t) \rangle$  by arc length  
- measure from  $t=0$

Sol.) First we compute the arc length function.

$$s(t) = \int_{t=0}^t \|\vec{r}'(a)\| da$$

$$\begin{aligned}\vec{r}'(t) &= \langle 3\cos(t), 2, -3\sin(t) \rangle \Rightarrow \|\vec{r}'(t)\| = \sqrt{(3\cos(t))^2 + 2^2 + (-3\sin(t))^2} \\ &= \sqrt{3^2 + 2^2} = \sqrt{13} \Rightarrow s(t) = \int_{a=0}^t \sqrt{13} da = [\sqrt{13}a]_0^t = \sqrt{13}(t-0)\end{aligned}$$

$$\therefore s(t) = \sqrt{13}t$$

- Thus the time is determined by  $t = \frac{s}{\sqrt{13}}$   
- Now we replace the parameter  $t$ :

$$\vec{r}(s) = \vec{r}(t(s)) = \vec{r}\left(\frac{s}{\sqrt{13}}\right) = \left\langle 3\sin\left(\frac{s}{\sqrt{13}}\right), \frac{2}{\sqrt{13}}s, 3\cos\left(\frac{s}{\sqrt{13}}\right) \right\rangle$$

NB: for  $\vec{r}(t)$  as above,

$$\vec{r}'(s) = \left\langle \frac{3}{\sqrt{13}}\cos\left(\frac{s}{\sqrt{13}}\right), \frac{2}{\sqrt{13}}, -\frac{3}{\sqrt{13}}\sin\left(\frac{s}{\sqrt{13}}\right) \right\rangle$$

so,

$$\begin{aligned}|\vec{r}'(s)| &= \sqrt{\left(\frac{3}{\sqrt{13}}\right)^2 \cos^2\left(\frac{s}{\sqrt{13}}\right) + \left(\frac{2}{\sqrt{13}}\right)^2 + \left(-\frac{3}{\sqrt{13}}\right)^2 \sin^2\left(\frac{s}{\sqrt{13}}\right)} \\ &= \sqrt{\frac{9}{13} + \frac{4}{13}} = \sqrt{\frac{13}{13}} = 1\end{aligned}$$

so,  $\vec{r}(s)$  is a unit-speed parameterization

## Physics nonsense

ex) Find the velocity and acceleration of the curve...

Soln:  $\vec{r}(t) = \langle 2^t, t^2, \ln(t+1) \rangle$  at time  $t=1$

velocity  $\vec{v}(t) = \vec{r}'(t) = \langle \ln(2)2^t, 2t, \frac{1}{t+1} \rangle$

$$= \langle \ln(2)2^t, 2t, (t+1)^{-1} \rangle$$

$$\vec{v}(1) = \langle \ln(2)2^1, 2 \cdot 1, (1+1)^{-1} \rangle = \langle 2\ln(2), 2, \frac{1}{2} \rangle$$

acceleration  $\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t) = \langle (\ln(2))^2 2^t, 2, -(t+1)^{-2} \rangle = \langle \ln(2)^2 2^t, 2, -(t+1)^{-2} \rangle$

$$\vec{a}(1) = \langle 2\ln(2)^2, 2, -\frac{1}{4} \rangle$$

ex) Find velocity and position of the curve satisfying

$$\vec{a}(t) = \langle 2, 0, 2t \rangle, \vec{v}(0) = \langle 3, -1, 0 \rangle, \vec{r}(0) = \langle 1, 0, 1 \rangle$$

Soln:  $\vec{v}(t) = \int \vec{a}(t) dt = \langle 2t, 0, t^2 \rangle + \vec{C}$

$$\Rightarrow \langle 3, -1, 0 \rangle = \vec{v}(0) = \langle 2(0), 0, 0 \rangle + \vec{C}$$

$$\therefore \vec{C} = \langle 3, -1, 0 \rangle - \langle 0, 0, 0 \rangle = \langle 3, -1, 0 \rangle$$

$$\therefore \vec{v}(t) = \langle 2t+3, -1, t^2 \rangle$$

so,  $\vec{r}(t) = \int \vec{v}(t) dt = \langle t^3 + 3t, -t, \frac{1}{3}t^3 \rangle + \vec{d}$

$$\Rightarrow \langle 1, 0, 1 \rangle = \vec{r}(0) = \langle 0^3 + 3(0), -0, \frac{0^3}{3} \rangle + \vec{d} = \vec{d}$$

$$\vec{r}(t) = \langle t^3 + 3t, -t, \frac{1}{3}t^3 \rangle + \vec{d}$$

$$\vec{r}(t) = \langle t^3 + 3t + 1, -t, \frac{1}{3}t^3 + 1 \rangle$$

←

ex) When is the particle w/ position function  $\vec{r}(t) = \langle t^2, 5t, 6t^2 - 16t \rangle$  moving the slowest?

Soln: Want to minimize the speed of  $\vec{r}(t)$   
i.e. Want to compute minimum of  $f(t) = |\vec{r}'(t)|$

Given  $\vec{r}(t) = \langle 2t, 5, 2t^2 - 16t \rangle$ , so  
 $f(t) = |\vec{r}'(t)| = \sqrt{(2t)^2 + 5^2 + (2t^2 - 16t)^2} = \sqrt{4t^2 + 25 + 4t^4 - 64t^3 + 256} = \sqrt{4t^2 - 64t + 281}$

→ first derivative test:

$$\begin{aligned} f'(t) &= \frac{1}{2}(8t^2 - 64t + 281)^{-\frac{1}{2}}(16t - 64) \\ &= \frac{8t - 32}{\sqrt{8t^2 - 64t + 281}} \end{aligned}$$

Note:  $(-64)^2 - 4 \cdot 8 \cdot 281 = 2^{12} - 2^5 \cdot 281 < 2^{12} - 2^5 \cdot 256$   
 $= 2^{12} - 2^5 \cdot 2^8 = 2^{12} - 2^{13} < 0$   
 $\therefore 8t^2 - 64t + 281 \neq 0$  for all  $t$ .

∴ the only critical points for  $f$  are  $8t - 32 = 0$ , i.e.  $t = 4$

Now,  $f'(0) = \frac{-32}{\sqrt{281}} < 0$  and  $f'(6) = \frac{8}{\sqrt{281}} > 0$

∴ the ~~given~~ particle is slowest at  $t = 4$ . □

Alternate Soln: if  $f(t) > 0$  for all  $t$ , then the critical points of  $(f(t))^2 = g(t)$  are precisely those of  $f(t)$ .

As before,  $f(t) = |\vec{r}'(t)| = \sqrt{8t^2 - 64t + 281}$

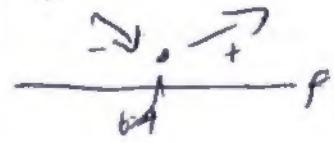
so,  $f(t) > 0$  for all  $t$  b/c  $(2t)^2 + 5^2 + (2t^2 - 16t)^2 \geq 5^2$  for all  $t$

∴ we can write  $g(t) = (f(t))^2 = 8t^2 - 64t + 281$

→  $g'(t) = 0$  iff  $16t - 64 = 0$  iff  $t = 4$

first derivative test

yields a minimum at  $t = 4$



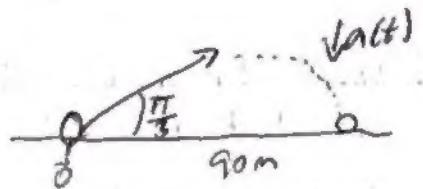
Ex) A ball is kicked from the ground at an angle of  $60^\circ$  if the ball lands 90m away, what was the initial speed of the ball?  
 $g = 9.81 \text{ m/s}^2$  towards earth

Sol:

$$\vec{r}(0) = \langle 0, 0 \rangle$$

$$\vec{a}(t) = \langle 0, -9.8 \rangle = \langle 0, -\frac{49}{5} \rangle$$

$$\vec{v}(0) = |\vec{v}(0)| \langle \cos\left(\frac{\pi}{3}\right), \sin\left(\frac{\pi}{3}\right) \rangle \rightarrow$$



$$\langle \cos\left(\frac{\pi}{3}\right), \sin\left(\frac{\pi}{3}\right) \rangle$$

$$\vec{v}(0) = |\vec{v}(0)| \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle, \quad \vec{r}(0) = \langle 0, 0 \rangle$$

$$\text{Want } C = |\vec{v}(0)|$$

$$\text{Now, } \vec{v}(t) = \int \vec{a}(t) dt = \left\langle 0, -\frac{49}{5}t + \beta \right\rangle$$

$$\text{we have, } C \left\langle \frac{1}{2}, \frac{\sqrt{3}}{2} \right\rangle = \vec{v}(0) = \left\langle 0, -\frac{49}{5}t + \beta \right\rangle$$

$$\alpha = \frac{C}{2} \quad \beta = \frac{\sqrt{3}}{2} C$$

$$\therefore \vec{v}(t) = \left\langle \frac{1}{2}C, -\frac{49}{5}t + \frac{\sqrt{3}}{2}C \right\rangle$$

$$\therefore \vec{r}(t) = \int \vec{v}(t) dt = \left\langle \frac{1}{2}Ct + \alpha, -\frac{49}{10}t^2 + \frac{\sqrt{3}}{2}C(t + \beta) \right\rangle$$

$$\text{Now } \langle 0, 0 \rangle = \vec{r}(0) = \left\langle \frac{1}{2}C(0) + \alpha, -\frac{49}{10}(0)^2 + \frac{\sqrt{3}}{2}C(0) + \beta \right\rangle = \langle \alpha, \beta \rangle$$

$$\vec{r}(t) = \left\langle \frac{1}{2}Ct, -\frac{49}{10}t^2 + \frac{\sqrt{3}}{2}Ct \right\rangle$$

$$\text{So at time } t = b: \left\{ \begin{array}{l} \frac{1}{2}Cb = 90 \\ -\frac{49}{10}b^2 + \frac{\sqrt{3}}{2}Cb = 0 \end{array} \right. \therefore \left\{ \begin{array}{l} -\frac{49}{10}b^2 + 90\sqrt{3} = 0 \\ C = \frac{180}{b} \end{array} \right.$$

$$\therefore \frac{900\sqrt{3}}{49} = b^2 \Rightarrow b = \pm \frac{30(3)^{\frac{1}{2}}}{7}, \text{ reject negative}$$

$$\text{So } C = \frac{180}{\frac{30(3)^{\frac{1}{2}}}{7}} - \frac{42}{3^{\frac{1}{2}}}$$